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Comparison Between the Convergence of Power and Bessel Series

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Abstract. We consider series defined by means of the Bessel functions, find their domains of convergence and study the behaviour of such series on the boundaries of these domains. Analogues of the classical theorems for the power series like Cauchy-Hadamard, Abel, as well as Fatou type theorems are proposed.

Keywords: Series in Bessel functions, Cauchy-Hadamard, Abel and Fatou type theorems.
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1. INTRODUCTION

Let $J_\nu(z)$ denote the Bessel function:

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k+\nu}}{k!\,\Gamma(k+\nu+1)}, \quad z \in \mathbb{C} \setminus (-\infty, 0], \quad \nu \in \mathbb{C}. \tag{1.1}$$

In the complex plane we consider series in the functions J_n $\nu = n = 0, 1, 2, \ldots$, called also Bessel coefficients, that is, the series of the kind

$$\sum_{n=0}^{\infty} a_n \widetilde{J}_n(z) \qquad (\widetilde{J}_n(z) = n! 2^n J_n(z)), \qquad a_n \in \mathbb{C}, \quad z \in \mathbb{C}. \tag{1.2}$$

We study where these series converge and where they do not, and moreover, where the convergence is uniform and where it is not. For the results discussed above, we need some suitable asymptotic formulae for the Bessel functions for "large" values of the indices. For example, the Bessel coefficients $J_n(z)$ have the following well-known representations (see e.g. [11, Ch.17, 17.81, p.375]):

$$J_n(z) = \left(\frac{z}{2}\right)^n (1 + \theta_n(z)) \frac{1}{n!}, \quad \theta_n(z) \to 0 \text{ as } n \to \infty \tag{1.3}$$

in the whole complex plane. The functions $\theta_n(z)$ are holomorphic for $z \in \mathbb{C}$ and moreover $\lim_{n\to\infty} \theta_n(z) = 0$ uniformly on each nonempty compact subset K of the plane $\mathbb{C}$. Considering explicitly $\theta_n(z)$, the result from (1.3), has been made sharper. Namely, as it is proved

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in [7], there exists a constant C, $0 < C < \infty$, such that for each $n \in \mathbb{N}_0$ and each $z \in K$, the following inequality holds

$$|\theta_n(z)| \le C/(n+1). \tag{1.4}$$

Remark 1.1 *The uniform convergence of $\theta_n(z)$ on the compact subsets of $\mathbb{C}$ follows from* (1.4)*, as well.*

Remark 1.2 *According to the asymptotic formula* (1.3)*, it follows that there exists a natural number N_0 such that the functions $\widetilde{J}_n(z)$ have no zeros for $n > N_0$, except for the origin.*

Remark 1.3 *Note that each of the functions $\widetilde{J}_n(z)$ $(n \in \mathbb{N}_0)$, being an entire function, no identically zero, has no more than finite number of zeros in the closed and bounded set $|z| \le R$ ([3], vol.1, ch. 3, §6, 6.1, p.305). Moreover, because of Remark 1.2, no more than finite number of these functions have some zeros, except for the origin.*

Together with the series (1.2) we consider the corresponding power series with the same coefficients:

$$\sum_{n=0}^{\infty} a_n z^n, \quad a_n \in \mathbb{C}. \tag{1.5}$$

Determining the disk of convergence of the series (1.2), we study its behaviour on the boundary of this domain, giving theorems of Cauchy-Hadamard, Abel and Fatou type, and compare the obtained results to the classical ones for the power series (1.5).

2. THEOREMS OF CAUCHY-HADAMARD AND ABEL TYPE

In the beginning we give a theorem of Cauchy-Hadamard type and a corollary for the series (1.2).

In what follows we use the notations $D(0;R)$ and $C(0;R)$ respectively for the open disk centered at the origin with radius R and its boundary, i.e.

$$D(0;R) = \{z : |z| < R,\ z \in \mathbb{C}\}, \quad C(0;R) = \partial D(0;R) = \{z : |z| = R,\ z \in \mathbb{C}\}.$$

Theorem 2.1 (of Cauchy-Hadamard type) *The domain of convergence of the series* (1.2) *is the circular domain $D(0;R)$ with a radius of convergence*

$$R = \left[\limsup_{n\to\infty} (|a_n|)^{1/n}\right]^{-1}. \tag{2.1}$$

More precisely, the series (1.2) *is absolutely convergent on the disk $D(0;R)$ and divergent on the domain $|z| > R$. The cases $R = 0$ and $R = \infty$ are included in the general case.*

Corollary 2.1.1 *Let the series* (1.2) *converges at the point* $z_0 \neq 0$. *Then it is absolutely convergent on the disk* $D(0;|z_0|)$. *Inside the disk* $D(0;R)$*, i.e. on each closed disk* $|z| \leq r$ *(*$r < R$*), the convergence is uniform.*

Proof. Indeed, since the considered series converges at the point $z_0 \neq 0$, then its radius of convergence R is a positive number, and moreover the point z_0 lies either in the disk $D(0;R)$ or on its boundary - the circle $C(0;R)$. That is why, the disk $D(0;|z_0|)$ is either a part of the domain of convergence or it coincides with it, whence the absolute convergence follows. To prove uniformity of the convergence inside the disk $D(0;R)$, it is sufficient to show that the series is uniformly convergent on each closed disk $|z| \leq r$ $(r < R)$. To this purpose, choosing a point ζ, $|\zeta| = \rho$, $r < \rho < R$ and considering the series (1.2), we estimate $|a_n \widetilde{J}_n(z)|$. First, mention that some of the values of $\widetilde{J}_n(\zeta)$, but only finite numbers of them, can be zero. Then, having in view (1.3), as well, there exists a number p, such that the expression $|a_n \widetilde{J}_n(z)|$ can be written as follows

$$|a_n \widetilde{J}_n(z)| = |a_n \widetilde{J}_n(\zeta)| \frac{|\widetilde{J}_n(z)|}{|\widetilde{J}_n(\zeta)|} = |a_n \widetilde{J}_n(\zeta)| \, \frac{|z^n||1+\theta_n(z)|}{|\zeta^n||1+\theta_n(\zeta)|} \leq |a_n \widetilde{J}_n(\zeta)| \, \frac{|1+\theta_n(z)|}{|1+\theta_n(\zeta)|}$$

for all $n > p$ and $|z| \leq r$.

Because of (1.4) and the relation $\lim\limits_{n\to\infty} \dfrac{1}{n+1} = 0$, we obtain the equalities $\lim\limits_{n\to\infty}(1 + \theta_n(z)) = 1$, $\lim\limits_{n\to\infty}(1+\theta_n(\zeta))^{-1} = 1$. Therefore, there exist numbers A and B such that $|1+\theta_n(z)||1+\theta_n(\zeta)|^{-1} \leq AB$ and hence $|a_n \widetilde{J}_n(z)| \leq AB|a_n \widetilde{J}_n(\zeta)|$, for all the values of $n > p$ and $|z| \leq r$. Since the series $\sum\limits_{n=0}^{\infty} a_n \widetilde{J}_n(\zeta)$ is absolutely convergent and by the Weierstrass criterium for the uniform convergence, the proof is completed. ∎

The very disk of convergence is not obligatory a domain of uniform convergence and the series may even be divergent on its boundary.

In this section we give some results for series of the kind (1.2) that refers to its domain of convergence and its behaviour "close" to the boundary of this domain. Such type of results are obtained also for series in other special functions, for example, for series in Laguerre and Hermite polynomials [10], and resp. by the author for systems of some other special functions of Fractional Calculus, which are fractional indices analogues of the Bessel functions and also multi-index Mittag-Leffler functions (in the sense of [2] and [1]), in the previous papers [4] - [6] and [8].

Let $z_0 \in \mathbb{C}$, $0 < R < \infty$, $|z_0| = R$, g_φ be an arbitrary angular domain with size $2\varphi < \pi$ and with a vertex at the point $z = z_0$, which is symmetric with respect to the straight line defined by the points 0 and z_0, and d_φ be the part of the angular domain g_φ, closed between the angle's arms and the arc of the circle with center at the point 0 and touching the arms of the angle. The following inequality can be verified inside d_φ:

$$|z - z_0| \cos\varphi < 2(|z_0| - |z|). \tag{2.2}$$

The next theorem refers to the uniform convergence on the set d_φ and the convergence at the point z_0, provided $|z| < R$ and $z \in g_\varphi$.

Theorem 2.2 (of Abel type) *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers, R be the radius defined by* (2.1)*, $F(z)$ be the sum of the series* (1.2) *on the domain $D(0;R)$ and this series converges at the point z_0 of the boundary of $D(0;R)$. Then:*

(I) *The following relation holds*

$$\lim_{z \to z_0} F(z) = \sum_{n=0}^{\infty} a_n \widetilde{J}_n(z_0),$$

provided $z \in D(0;R) \cap g_{\varphi}$.

(II) *The series* (1.2) *is uniformly convergent on the domain d_{φ}.*

The proofs of Theorems 2.1 and 2.2, excluding the uniformity, are given in [4].

Proof. To prove the uniform convergence we use the inequality (2.2) that is the crucial point of the proof.

So, let $z \in d_{\varphi}$. Setting

$$S_k(z) = \sum_{n=0}^{k} a_n \widetilde{J}_n(z), \quad S_k(z_0) = \sum_{n=0}^{k} a_n \widetilde{J}_n(z_0), \quad \lim_{k \to \infty} S_k(z_0) = s, \tag{2.3}$$

$$\beta_n = S_n(z_0) - s, \quad \beta_n - \beta_{n-1} = a_n \widetilde{J}_n(z_0),$$

we obtain

$$S_{k+p}(z) - S_k(z) = \sum_{n=0}^{k+p} a_n \widetilde{J}_n(z) - \sum_{n=0}^{k} a_n \widetilde{J}_n(z) = \sum_{n=k+1}^{k+p} a_n \widetilde{J}_n(z).$$

According to Remark 1.2, there exists a natural number N_0 such that $\widetilde{J}_n(z_0) \neq 0$ when $n > N_0$. Let $k > N_0$ and $p > 0$. Then, using the denotation

$$\gamma_n(z;z_0) = \widetilde{J}_n(z)/\widetilde{J}_n(z_0),$$

we can write the difference $S_{k+p}(z) - S_k(z)$ as follows:

$$S_{k+p}(z) - S_k(z) = \sum_{n=k+1}^{k+p} a_n \widetilde{J}_n(z_0) \frac{\widetilde{J}_n(z)}{\widetilde{J}_n(z_0)} = \sum_{n=k+1}^{k+p} a_n \widetilde{J}_n(z_0) \gamma_n(z;z_0).$$

Now, by the Abel transformation (see in [3], Vol.1, Ch.1, p.32, 3.4:7), we obtain subsequently:

$$S_{k+p}(z) - S_k(z) = \sum_{n=k+1}^{k+p} (\beta_n - \beta_{n-1}) \gamma_n(z;z_0)$$

$$= \beta_{k+p}\gamma_{k+p}(z;z_0) - \beta_k \gamma_{k+1}(z;z_0) - \sum_{n=k+1}^{k+p-1} \beta_n (\gamma_{n+1}(z;z_0) - \gamma_n(z;z_0)),$$

$$S_{k+p}(z) - S_k(z) = (S_{k+p}(z_0) - s)\gamma_{k+p}(z;z_0) - (S_k(z_0) - s)\gamma_{k+1}(z;z_0)$$
$$+ \sum_{n=k+1}^{k+p-1} (S_n(z_0) - s) \times \left(\frac{\widetilde{J}_n(z)}{\widetilde{J}_n(z_0)} - \frac{\widetilde{J}_{n+1}(z)}{\widetilde{J}_{n+1}(z_0)} \right).$$

So, using last relation, we are going to estimate the module of the difference $S_{k+p}(z) - S_k(z)$ as follows:

$$\begin{aligned}|S_{k+p}(z)-S_k(z)| &\le |S_{k+p}(z_0)-s||\gamma_{k+p}(z;z_0)|+|S_k(z_0)-s||\gamma_{k+1}(z;z_0)| \\ &+\sum_{n=k+1}^{k+p-1}|S_n(z_0)-s|\times\left|\frac{\widetilde{J}_n(z)}{\widetilde{J}_n(z_0)}-\frac{\widetilde{J}_{n+1}(z)}{\widetilde{J}_{n+1}(z_0)}\right|. \end{aligned}\tag{2.4}$$

Because of (1.4) and the relations $\lim_{n\to\infty}\frac{1}{n+1}=0$, $\lim_{n\to\infty}(1+\theta_n(z_0))^{-1}=1$, there exist numbers A and $N_1>N_0$ such that $|1+\theta_n(z)|\le A/2$ for all the natural values of n and $|1+\theta_n(\zeta)|^{-1}<2$ for $n>N_1$, whence

$$|\gamma_n(z,z_0)|\le A \quad \text{for } n>N_1. \tag{2.5}$$

Further, setting

$$j_n(z,z_0)=\frac{\widetilde{J}_n(z)}{\widetilde{J}_n(z_0)}-\frac{\widetilde{J}_{n+1}(z)}{\widetilde{J}_{n+1}(z_0)}=\frac{z^n}{z_0^n}\times\left(\frac{1+\theta_n(z)}{1+\theta_n(z_0)}-\frac{z}{z_0}\times\frac{1+\theta_{n+1}(z)}{1+\theta_{n+1}(z_0)}\right)$$

and observing that $j_n(z_0,z_0)=0$, we apply the Schwarz lemma, named after Hermann Amandus Schwarz, for $j_n(z,z_0)$. Thus, we get that there exists a constant C:

$$|j_n(z,z_0)|=|\widetilde{J}_n(z)/\widetilde{J}_n(z_0)-\widetilde{J}_{n+1}(z)/\widetilde{J}_{n+1}(z_0)|\le C|z-z_0||z/z_0|^n,$$

whence, and in accordance with (2.2):

$$\sum_{n=k+1}^{k+p+1}|j_n(z,z_0)|\le\sum_{n=0}^{\infty}C|z-z_0||z/z_0|^n=C|z_0|\times\frac{|z-z_0|}{|z_0|-|z|}<\frac{2C|z_0|}{\cos\varphi}. \tag{2.6}$$

Let ε be an arbitrary positive number. Taking in view the third of the relations (2.3), we can confirm that there exists a positive number $N_2>N_0$ so large that

$$|S_n(z_0)-s|<\min\left(\frac{\varepsilon}{3A},\frac{\varepsilon\cos\varphi}{6C|z_0|}\right)\quad \text{for } n>N_2. \tag{2.7}$$

Now, let $N=N(\varepsilon)=\max(N_1,N_2)$ and $k>N$. Therefore (2.4)–(2.6) give

$$|S_{k+p}(z)-S_k(z)|<\frac{2\varepsilon}{3}+\frac{\varepsilon\cos\varphi}{6C|z_0|}\times\sum_{n=k+1}^{k+p+1}|j_n(z,z_0)|<\frac{2\varepsilon}{3}+\frac{\varepsilon\cos\varphi}{6C|z_0|}\times\frac{2C|z_0|}{\cos\varphi}=\varepsilon,$$

that completes the proof of the theorem for the considered series. ■

Remark 2.1 *If the series* (1.2) *has a finite and non-zero radius of convergence R, it converges at the point* $z_0\in C(0;R)$ *and F is the holomorphic function defined by this series in its domain of convergence* $D(0;R)$*, then by the Theorem 2.2 it follows that*

$$\lim_{z\to z_0,\, z\in d_\varphi}F(z)=F(z_0),$$

i.e. the restriction of the function F to each set of the kind d_φ *is continuous at the point* z_0.

3. FATOU TYPE THEOREM

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers with $\limsup_{n\to\infty} (|a_n|)^{1/n} = R^{-1}$, $0 < R < \infty$ and $f(z)$ be the sum of the power series $\sum_{n=0}^{\infty} a_n z^n$ on the open disk $D(0;R)$, i.e.

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in D(0;R).$$

Definition 3.1 *A point $z_0 \in \partial D(0;R)$ is called regular for the function f if there exist a neighbourhood $U(z_0;\rho)$ and a function $f_{z_0}^* \in \mathscr{H}(U(z_0;\rho))$ (the space of complex-valued functions, holomorphic in the set $U(z_0;\rho)$), such that $f_{z_0}^*(z) = f(z)$ for $z \in U(z_0;\rho) \cap D(0;R)$.*

By this definition it follows that the set of regular points of the power series is an open subset of the circle $C(0;R) = \partial D(0;R)$ with respect to the relative topology on $\partial D(0;R)$, i.e. the topology induced by that of $\mathbb{C}$.

In general, there is no relation between the convergence (divergence) of a power series at points on the boundary of its disk of convergence and the regularity (singularity) of its sum of such points. For example, the power series $\sum_{n=0}^{\infty} z^n$ is divergent at each point of the unit circle $C(0;1)$ regardless of the fact that all the points of this circle, except for $z = 1$, are regular for its sum. The series $\sum_{n=1}^{\infty} n^{-2} z^n$ is (absolutely) convergent at each point of the circle $C(0;1)$, but nevertheless one of them, namely $z = 1$, is a singular (i.e. not regular) for its sum. But under additional conditions on the sequence $\{a_n\}_{n=0}^{\infty}$, such a relation does exists (see for details [3], Vol.1, Ch. 3, §7, 7.3, p.357), namely, if the coefficients of the power series with the unit disk of convergence tend to the zero, i.e. $\lim_{n\to\infty} a_n = 0$, then the power series converges, even uniformly, on each arc of the unit circle, all points of which (including the ends of the arc) are regular for the sum of the series.

Propositions referring to the properties discussed above have been established also for series in the Laguerre and Hermite polynomilas, as well as in Mittag-Leffler systems (see e.g. [10], resp. [9]). Here we give such type of theorem for the Bessel systems as follows.

Theorem 3.1 (of Fatou type) *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers satisfying the conditions $\lim_{n\to\infty} a_n = 0$, $\limsup_{n\to\infty} (|a_n|)^{1/n} = 1$, and $F(z)$ be the sum of the series* (1.2) *on the unit disk $D(0;1)$, i.e.*

$$F(z) = \sum_{n=0}^{\infty} a_n \widetilde{J}_n(z), \quad z \in D(0;1).$$

Let σ be an arbitrary arc of the unit circle $C(0;1)$ with all its points (including the ends) regular to the function F. Then the series (1.2) *converges, even uniformly, on the arc σ.*

Proof. Since all the points of the arc σ are regular to the function $F(z)$ there exists a region $G \supset \sigma$ where the function F can be continued. Denoting $\widetilde{G} = G \cup D(0;1)$, we define the function ψ in the region $\widetilde{G}$ by the equality

$$\psi(z) = F(z), \quad z \in D(0;1).$$

More precisely, it means that ψ is a single valued analytical continuation of F in the domain $\widetilde{G}$.

Let $\rho > 0$ be the distance between the boundary $\partial\widetilde{G}$ of the region $\widetilde{G}$ and the arc σ ($\partial\widetilde{G}$ contains a part of the unit circle $|z| = 1$), and take the points ζ_1, ζ_2:

$$\zeta_1, \zeta_2 \notin \sigma, \quad |\zeta_1| = |\zeta_2| = 1,$$

such that the distances between each of the points ζ_1, ζ_2 and the respective closer end of the arc σ are equal to $\rho/2$, and

$$z_1 = \zeta_1(1+\rho/2), \quad z_2 = \zeta_2(1+\rho/2).$$

Define the auxiliary function

$$\varphi_n(z) = \psi(z) - \sum_{k=0}^{n} a_k \widetilde{J}_k(z) \tag{3.1}$$

and note that, according to Remark 1.2, there exists a natural number N_0 such that $\widetilde{J}_n(z_0) \neq 0$ when $n > N_0$. Now, letting $n \geq N_0$, we introduce the notation

$$\omega_n(z) = \frac{\varphi_n(z)}{\widetilde{J}_{n+1}(z)}(z-\zeta_1)(z-\zeta_2). \tag{3.2}$$

In order to prove that the sequence $\left\{ \sum_{k=0}^{n} a_k \widetilde{J}_k(z) \right\}$ is uniformly convergent on the arc σ, it is sufficiently to show that the sequence $\{\omega_n(z)\}_{n=N_0}^{\infty}$ tends uniformly to zero on the boundary $\partial\Delta$ of the sector $\Delta = Oz_1z_2$ which is a compact set and after that estimate $\varphi_n(z)$ on the arc σ.

To this end, we come back to (1.4). Just mention that since $\lim_{n\to\infty} \frac{1}{n+1} = 0$, there exist numbers C and $\widetilde{N} > N_0$ such that $|1+\theta_n(z)| \leq C/2$ for all the values of $n \in \mathbb{N}$ and $1/2 \leq |1+\theta_n(z)| \leq 2$ for $n > \widetilde{N}$ on an arbitrary compact subset of $\mathbb{C}$.

Now, taking $\varepsilon > 0$ and setting

$$R = 1+\rho/2, \quad \varepsilon_1 = \frac{\varepsilon\rho^3}{8(8CR^2+\rho)}, \quad M = \max_{z\in[\Delta]} |\psi(z)| \qquad ([\Delta] = \Delta \cup \partial\Delta),$$

we have to consider four cases as follows.

1) First, let $z \in (O,\zeta_1) \cup (O,\zeta_2) \subset D(0;1)$.

In the unit disk, according to (3.1), we have consecutively:

$$\omega_n(z) = \sum_{k=0}^{\infty} a_{n+k+1} \frac{\widetilde{J}_{n+k+1}(z)}{\widetilde{J}_{n+1}(z)} (z-\zeta_1)(z-\zeta_2),$$

$$\omega_n(z) = \sum_{k=0}^{\infty} a_{n+k+1}\, z^k\, \frac{(1+\theta_{n+k+1}(z))}{(1+\theta_{n+1}(z))} (z-\zeta_1)(z-\zeta_2). \tag{3.3}$$

Since $a_n \to 0$, there exists a number $N_1 = N_1(\varepsilon_1) > \widetilde{N}$, such that

$$|\omega_n(z)| < \varepsilon_1 \sum_{k=0}^{\infty} |z|^k \left| \frac{(1+\theta_{n+k+1}(z))}{(1+\theta_{n+1}(z))} \right| |(z-\zeta_1)||(z-\zeta_2)|$$

$$< 2C\varepsilon_1 \sum_{k=0}^{\infty} |z|^k (1-|z|) = 2C\varepsilon_1$$

for $n > N_1$, i.e.

$$|\omega_n(z)| < 2C\varepsilon_1. \tag{3.4}$$

2) $z \in (\zeta_1, z_1) \cup (\zeta_2, z_2)$.

In this case $|z-\zeta_1| = |z|-1$, $|z-\zeta_2| \le |z| + |\zeta_2| < 2R$, and taking into account (1.3) and (3.1) we can write the following inequalities for the absolute value of $\omega_n(z)$

$$\omega_n(z) = \frac{\psi(z) - \sum\limits_{k=0}^{n} a_k\, z^k\, (1+\theta_k(z))}{z^{n+1}(1+\theta_{n+1}(z))} (z-\zeta_1)(z-\zeta_2),$$

namely

$$|\omega_n(z)| \le \frac{M + \sum\limits_{k=0}^{n} |a_k||z|^k|(1+\theta_k(z))|}{|z|^{n+1}|(1+\theta_{n+1}(z))|}\, 2R(|z|-1)$$

$$< 2R \left(2M + \sum_{k=0}^{N_1} C|a_k|R^k \right) \frac{(|z|-1)}{|z|^{n+1}} + 2\varepsilon_1 RC\, \frac{(|z|-1)}{|z|^{n+1}} \sum_{k=N_1+1}^{n} |z|^k .$$

Furthermore, having in mind that, on the one hand:

$$\frac{(|z|-1)}{|z|^{n+1}} < \frac{(|z|-1)}{|z|^{n+1}-1} = \frac{1}{|z|^n + \cdots + 1} < \frac{1}{n+1},$$

and on the other hand:

$$\sum_{k=N_1+1}^{n} |z|^k = \frac{|z|^{n+1} - |z|^{N_1+1}}{(|z|-1)} < \frac{|z|^{n+1}}{(|z|-1)},$$

we conclude that

$$|\omega_n(z)| < \frac{2R}{n+1} \left(2M + \sum_{k=0}^{N_1} C|a_k|R^k \right) + 2\varepsilon_1 RC.$$

Then, since $n^{-1} \to 0$, there exists a number $N_2 = N_2(\varepsilon_1) > N_1$ such that

$$\frac{2R}{n+1}\left(2M + \sum_{k=0}^{N_1} C|a_k|R^k\right) < \varepsilon_1$$

for $n > N_2$, i.e.

$$|\omega_n(z)| < (1+2RC)\varepsilon_1. \tag{3.5}$$

3) z belongs to the arc $\widehat{z_1 z_2}$ (including the ends).

Then $|z-\zeta_1| < 2R$, $|z-\zeta_2| < 2R$ and hence

$$|\omega_n(z)| < \frac{4R^2\left(2M + \sum_{k=0}^{n} C|a_k|R^k\right)}{R^{n+1}} < \frac{4\left(2M + \sum_{k=0}^{N_1} C|a_k|R^k\right)}{R^{n-1}} + \frac{4\varepsilon_1 C\left(\sum_{k=N_1+1}^{n} R^k\right)}{R^{n-1}}$$

$$< \frac{4\left(2M + \sum_{k=0}^{N_1} C|a_k|R^k\right)}{R^{n-1}} + \frac{8\varepsilon_1 CR^2}{\rho}.$$

Since $R^{-n} \to 0$, there exists a number $N_3 = N_3(\varepsilon_1) > N_1$, such that

$$|\omega_n(z)| < \left(\frac{8CR^2}{\rho} + 1\right)\varepsilon_1 \tag{3.6}$$

for $n > N_3$.

4) $z \in \{O, \zeta_1, \zeta_2\}$.

In this case we have $\omega_n(0) = a_{n+1}\zeta_1\zeta_2$, whence $|\omega_n(0)| = |a_{n+1}| < \varepsilon_1$ for $n > N_1$, and $\omega_n(\zeta_{1,2}) = 0$.

Let $N = \max\{N_1, N_2, N_3\}$ and $n > N$, then having in view the inequalities (3.4) – (3.6), we can write on the boundary of the region Δ:

$$|\omega_n(z)| < \max\left(2C\varepsilon_1, (2RC+1)\varepsilon_1, \left(\frac{8CR^2}{\rho} + 1\right)\varepsilon_1\right) = \left(\frac{8CR^2}{\rho} + 1\right)\varepsilon_1.$$

Hence according to the principle of the maximum of the modulus

$$|\omega_n(z)| < \left(\frac{8CR^2 + \rho}{\rho}\right)\varepsilon_1, \quad z \in \sigma. \tag{3.7}$$

Eventually, according to (1.3), (3.1) and (3.3), since $|z| = 1$ on the arc σ,

$$|\omega_n(z)| = \frac{\left|\psi(z) - \sum_{k=0}^{n} a_k \widetilde{J}_k(z)\right|}{|z^{n+1}||1+\theta_{n+1}(z)|}|z-\zeta_1||z-\zeta_2| > \frac{1}{2}\cdot\frac{\rho^2}{4}\left|\psi(z) - \sum_{k=0}^{n} a_k \widetilde{J}_k(z)\right|, \tag{3.8}$$

whence applying the inequality (3.7), the relation (3.8) yields to

$$\left|\psi(z) - \sum_{k=0}^{n} a_k \widetilde{J}_k(z)\right| < \frac{8}{\rho^2}|\omega_n(z)| < \frac{8\varepsilon_1}{\rho^3}\left(8CR^2 + \rho\right) = \varepsilon, \quad z \in \sigma,$$

that proves the theorem. ∎

4. CONCLUSION

We note that the results obtained for the series (1.2) are the same as these for the power series (1.5). As it is well seen, they have one and the same radius of convergence R, and both are absolutely and uniformly convergent on each closed disk $|z| \leq r$ $(r < R)$. More precisely, if each of them converges at the point z_0 of the boundary of $D(0;R)$, then the theorems of Abel type hold for both series in one and the same angular domain. Finally, if $\{a_n\}_{n=0}^{\infty}$ is a sequence of complex numbers satisfying the conditions $\lim\limits_{n\to\infty} a_n = 0$, $\limsup\limits_{n\to\infty} (\,|a_n|\,)^{1/n} = 1$, and all the points (including the ends) of the arc σ of the unit circle $|z| = 1$ are regular to the sum of both considered series, then the series (1.2) and (1.5) converge even uniformly, on the arc σ.

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